DATE OF EXAM: 22-09-2008
SUBJECT NAME: Advanced Functional Analysis - MIDTERM Exam - M. Math 2008-09

---

1. **Problem:** Let X be a metrizable TVS and Y a TVS. Let $T : X \to Y$ be a linear map such that for every sequence $\{x_n\}_{n\geq 1} \subset X, x_n \to 0$ implies, $\{T(x_n)\}_{n\geq 1}$ is a Cauchy sequence. Show that T is continuous.

   Solution: We know that a Cauchy sequence in a TVS is bounded. It is given that for every sequence $\{x_n\}_{n\geq 1} \subset X, x_n \to 0$ implies $\{T(x_n)\}_{n\geq 1}$ is a Cauchy sequence. Hence $\{T(x_n)\}_{n\geq 1}$ is bounded. Thus T is continuous. (See, Thm 1.32 in Rudin FA).

   □

2. **Problem:** Let X be a LCTVS such that every closed and bounded set is compact. Let $F \subset X$ be a totally bounded set. Show that the closed convex hull, $\mathrm{CO}^-(\Gamma F)$ is a compact set, where Γ is the unit circle.

   Solution: In a LCTVS the convex hull of a totally bounded set is totally bounded (See Thm. 3.20 Rudin FA). Here it is given that F is totally bounded in a LCTVS X. First we shall show that ΓF is totally bounded, where Γ is the unit circle. Take a nbd $V \in \mathcal{N}_0$. Then there exists a balanced nbd $U \in \mathcal{N}_0$ such that $U + U \subset V$. Since F is totally bounded, for $U \in \mathcal{N}_0$ there exists a finite set E such that $F \subset E + U$. This implies that

   $$\Gamma F \subset \Gamma E + \Gamma U \subset \Gamma E + U,$$

   since U is balanced. Note that for the finite set E, $\Gamma E = \cup_{x\in E}\Gamma\{x\}$ is totally bounded. Thus for the nbd $U \in \mathcal{N}_0$ there exists a finite set $E^{'}$, such that $\Gamma E \subset E^{'} + U$. As a result we have

   $$\Gamma F \subset E^{'} + U + U \subset E^{'} + V,$$

   which proves that ΓF is totally bounded. Hence we have $\mathrm{CO}(\Gamma F)$ is totally bounded and hence bounded. So $\mathrm{CO}^-(\Gamma F)$ is a closed and bounded set in X. Hence from assumption it is compact.

   □

3. **Problem:** Consider $\Lambda_m : l^2 \to \mathbb{C}$ defined by $\Lambda_m(x) = \sum_1^m n^2 x(n)$. Let $x_n = \frac{e_n}{n}$. Show that each $\Lambda_m(K)$ is a bounded set but $\{\Lambda_m(K)\}_{m\geq 1}$ is not uniformly bounded.

   Solution: To show that $K = \{x_n\}_{n\geq 1} \cup \{0\}$ is compact, let $K \subset \cup_{\alpha\in I} V_\alpha$, an arbitrary open cover of K. Then there exists an open set V_α from the arbitrary collection which contains zero. Since $x_n = \frac{e_n}{n} \to 0$ in l_2, we shall have $x_n \in V_\alpha$ except finitely many points. Those finitely many x_n can be covered in finitely many open sets. This proves that K can be covered by finitely many open sets and hence it is compact.

   For the other part note that

   $$\begin{aligned} \Lambda_m(x_n) \quad &= m \ \text{ if } \ n \leq m \\ &= 0 \ \text{ if } \ n > m. \end{aligned}$$

   This shows that $\Lambda_m(K)$ is a bounded set but $\{\Lambda_m(K)\}_{m\geq 1}$ is not uniformly bounded. □

4. **Problem:** Let X be a LCTVS. Let $A \subset X$ be a balanced, closed, convex set. Show that ${}^{o}(A^{o}) = A$.

   Solution: From Bipolar Thm (See Thm 1.8 in Conway FA) we know that ${}^{o}(A^{o})$ is the closed convex balanced hull of A. Since A is closed, convex, balanced from assumption we have

$$^{o}(A^{o}) = A.$$

$\square$

5. **Problem:** Let X be a TVS and A, B two compact, convex sets. Show that the convex hull $\mathrm{CO}(A \cup B)$ is compact.

   Solution: See Thm 3.20 in Rudin FA. $\square$

6. **Problem:** Let $l^1 = \{\{\alpha_n\}_{n\geq 1} \subset \mathbb{R} : \sum |\alpha_n| < \infty\}$. Show that the extreme points of the closed unit ball are precisely $\{\pm e_n\}_{n\geq 1}$.

   Solution: Let $K = \{\{\alpha_n\}_{n\geq 1} \subset \mathbb{R} : \sum |\alpha_n| \leq 1\} \subset l^1$. To show that the extreme points of K are precisely $\{\pm e_n\}_{n\geq 1}$. It is easy to see that $\mathrm{ext}(K) \subset \{\{\alpha_n\}_{n\geq 1} \subset \mathbb{R} : \sum |\alpha_n| = 1\}$. As if $x \in \mathrm{ext}(K)$ and $\epsilon = \| x \| < 1$, then

$$x = \epsilon(\frac{x}{\epsilon}) + (1-\epsilon)0.$$

Let $\{\alpha_n\} \in \mathrm{ext}(K)$ and $\{\alpha_n\} \neq \{\pm e_m\}$, for any m. Then $|\alpha_n| < 1$ for all $n \in \mathbb{N}$. Since $\alpha_n \to 0$, there exists $n_0 \in \mathbb{N}$ such that $|\alpha_n| < 1/2$ for all $n \geq n_0$. Let $\epsilon > 0$ be such that,

$$1 > \epsilon > \sup\{|\alpha_1|, \cdots |\alpha_{n_0-1}|, 1/2\}.$$

Then $1 > \epsilon > |\alpha_n|, \forall n \in \mathbb{N}$. Now,

$$\{\alpha_n\} = \epsilon\frac{\{\alpha_n\}}{\epsilon} + (1-\epsilon)0.$$

This shows that $\{\alpha_n\}$ is not an extreme point, a contradiction. Conversely, to prove $\{\pm e_n\}$ are extreme points of K, let

$$\{e_n\} = \frac{\{a_m\} + \{b_m\}}{2},$$

where $\{a_m\}, \{b_m\} \in K$. This shows that $1 = \frac{a_n+b_n}{2}$. Hence $a_n = b_n = 1$ and $a_m = b_m = 0; \forall m \neq n$. Hence $\{a_n\} = \{e_n\}, \{b_n\} = \{e_n\}$. Thus $\{e_n\}$ are extreme points of K. Similarly for $\{-e_n\}$.

$\square$

7. **Problem:** Let K be a compact, Hausdorff space. Let $C(K)^*_1$ be the closed unit ball of $C(K)^*$, equipped with the weak$^*-$ topology. Let $\mathcal{P}$ denote the set of all probability measures. Show that $C(K)^*_1 = \mathrm{CO}^-(\Gamma\mathcal{P})$, where the closure is taken in the weak$^*-$ topology.

   Solution: The closed unit ball $C(K)^*_1$ is compact in $C(K)^*$, with the weak$^*-$ topology. From Krein-Milman theorem we have

$$C(K)^*_1 = \mathrm{CO}^-(\mathrm{ext}(C(K)^*_1)).$$

Now,

$$\mathrm{ext}(C(K)^*_1) = \{\alpha\delta_x : |\alpha| = 1, x \in K\},$$

and

$$\text{ext}(\mathcal{P}) = \{\delta_x : x \in K\}.$$

(See Thm 8.4 in Conway FA).
Thus, $\Gamma\text{ext}(\mathcal{P}) = \Gamma\{\delta_x : x \in K\} = \text{ext}(C(K)_1^*)$ As a result we have

$$\begin{aligned} C(K)_1^* = \text{CO}^-(\text{ext}(C(K)_1^*)) &= \text{CO}^-(\Gamma\text{ext}(\mathcal{P})) \\ &\subset \text{CO}^-(\Gamma\mathcal{P}) \\ &\subset \text{CO}^-(C(K)_1^*) \quad \text{(From Reisz Repn Thm)} \\ &= C(K)_1^*. \quad \text{(Since } \ C(K)_1^* \ \text{ is closed and convex)} \end{aligned}$$

Thus, $C(K)_1^* = \text{CO}^-(\Gamma\mathcal{P})$.

$\square$